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On the relationship between Kolmogorov's and generalized structure functions in the inertial subrange of developed turbulence

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Abstract. A relationship between Kolmogorov's $D_{Kp}(r)$ and generalized $D_{Gp}(r)$ structure functions of odd orders is suggested and tested. The approach is based on scaling considerations for a two-scale structure function $D_{2Gp}(l, r)$ introduced in this paper. It is shown that for r from the inertial subrange, the relationship between $D_{Kp}(r)$ and $D_{Gp}(r)$ is linear which implies identity between scaling exponents obtained from $D_{Kp}(r)$ and $D_{Gp}(r)$. We also show that $D_{2Gp}(l, r)$ provides more information about turbulence structure than $D_{Kp}(r)$ or $D_{Gp}(r)$ which are special cases of $D_{2Gp}(l, r)$.

1. Introduction

In 1941 Kolmogorov introduced the structure function as a tool to study small-scale turbulence (Monin and Yaglom 1975, Frisch 1995). He defined the longitudinal structure function as

$$D_{Kp}(r) = \langle \Delta u(r)^p \rangle \tag{1}$$

where $\Delta u(r) = [u(x + r) - u(x)]$ is the velocity increment between two points lying on the x-axis and separated by distance r, p is the order of the structure function, and angular brackets define averaging over many points. For r from the inertial subrange of scales (i.e., much less than the external flow scale and much larger than the scale where dissipation occurs) Kolmogorov's initial theory (referred to hereafter as K41) predicts the following relationship for the velocity structure function:

$$D_{Kp}(r) = c_p \bar{\varepsilon}^{p/3} r^{\xi(p)} \tag{2}$$

where the constants c_p are presumed to be 'universal', $\bar{\varepsilon}$ is the mean energy dissipation, and $\xi(p) = p/3$. The deviation of measured exponents $\xi(p)$ from $p/3$ (especially profound for large p) has inspired revisions of K41 which incorporate intermittency in the velocity, vorticity, and/or dissipation fields (Vainshtein *et al* 1994, Frisch 1995). The first revision was suggested by Kolmogorov himself in 1962 and now it is known as the refined Kolmogorov's hypotheses or as K62 (Monin and Yaglom 1975, Frisch 1995). In general, the revised expression for the velocity structure function in the inertial subrange can be presented as

$$D_{Kp}(r) = c_p^* \bar{\varepsilon}^{p/3} L_0^{p/3-\xi(p)} r^{\xi(p)} \tag{3}$$

where the definition of exponents $\xi(p)$ depends on the particular intermittency model, L_0 is the external turbulence scale (e.g., integral scale), and the c_p^* is a new set of constants related to c_p as $c_p = c_p^*(L_0/r)^{p/3-\xi(p)}$. Relationships (2) and (3) are derived from phenomenological

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considerations except for the third-order longitudinal structure function which Kolmogorov derived rigorously from the Navier–Stokes equation, i.e.

$$D_{K3}(r) = -\tfrac{4}{5}\bar{\varepsilon}r. \tag{4}$$

Equation (4) is widely used as a kind of 'boundary condition' on theories of turbulence (to be acceptable, such theories must satisfy (4), or explicitly violate the assumptions made to derive it, Frisch (1995)). Until recently, Kolmogorov's structure functions have been used directly as tools for testing various intermittency models. Recently, Benzi *et al* (1993a, b) have introduced a new concept known as extended self-similarity (ESS). According to this concept, the scaling is significantly better if data are presented as

$$D_{Kp}(r) = m_p|D_{K3}(r)|^{\xi^*(p)} \tag{5}$$

where the m_p are constants independent of r, $D_{K3}(r) = \langle \Delta u(r)^3 \rangle$, and $\xi^*(p)$ is assumed to be equal to $\xi(p)$. Benzi *et al* (1993a, b) have also suggested that relationship (5) be valid for moments of the absolute values of velocity increments, i.e.

$$D_{Gp}(r) = m_p^* D_{G3}^{\xi^{**}(p)}(r) = m_p^{**}|D_{K3}(r)|^{\xi^{***}(p)} \tag{6}$$

where $D_{Gp}(r) = \langle |\Delta u(r)|^p \rangle$ are the so-called generalized structure functions, m_p^* and m_p^{**} are new sets of constants, and $\xi^*(p) = \xi^{**}(p) = \xi^{***}(p)$ are assumed. The latter equalities are only valid if $|D_{Kp}(r)| \propto D_{Gp}(r)$, which is automatically true for even integer orders, but for odd p the situation is unclear. To support the validity of (6) with related assumptions, Benzi *et al* (1993b) presented an empirical relationship between $|D_{K3}(r)|$ and $D_{G3}(r)$ which appeared to be linear for a wide range of scales and for various flows (their analysis was restricted to the third-order structure functions only). However, Herweijer (1995), and Sreenivasan and Dhruva (1998) have presented somewhat different results, claiming that the relationship between $|D_{K3}(r)|$ and $D_{G3}(r)$ is slightly nonlinear (i.e. $D_{G3}(r) \propto |D_{K3}(r)|^{1.05}$) which automatically means invalidity of the equality $\xi^*(p) = \xi^{**}(p) = \xi^{***}(p)$. Also, Vainshtein and Sreenivasan (1994) found empirically that $\xi^*(p) \geqslant \xi^{**}(p)$ for odd p, while Arneodo *et al* (1996) pointed out that the difference between exponents $\xi^*(p)$, $\xi^{**}(p)$, and $\xi^{***}(p)$ for $p = 6$ is about 10%.

The initial reason to introduce $D_{Gp}(r)$ for scaling considerations was technical rather than theoretical. An example of the typical justification is 'that it is statistically more stable than $D_{Kp}(r)$' (Benzi *et al* 1993b, p 277). However, in a later work, Vainshtein *et al* (1994) have used the generalized structure functions $D_{Gp}(r)$ as a theoretical tool to build a general model which connects exponents for the structure functions with multifractal exponents for the vorticity and dissipation fields. In their derivation, Vainshtein *et al* (1994) have also assumed that the equality $\xi^*(p) = \xi^{**}(p) = \xi^{***}(p)$ is valid (although the authors recognized the fact that for odd p this equality still awaits justification). At present, Kolmogorov's and generalized structure functions are often used (rather implicitly) as interchangeable functions although Kolmogorov's initial ideas and derivations (e.g., relationships (2)–(4)) relate only to $D_{Kp}(r)$, and not to $D_{Gp}(r)$. In this paper we present an attempt to clarify a relationship between $D_{Kp}(r)$ and $D_{Gp}(r)$ and to identify the prefactor in this relationship.

2. Scaling considerations

Let us introduce a two-scale generalized structure function $D_{2Gp}(l, r)$:

$$D_{2Gp}(l, r) = \frac{1}{Nn} \sum_{j}^{N(l)} \left| \sum_{i}^{n(l)} \Delta u(r)_i^p \right|_j \tag{7}$$

where $N(l)$ is the number of equal and non-overlapping subsets of the length l, $l \geqslant r$, $n(l) = l/r$ is the number of velocity increments within a subset, and $N(l)n(l)$ is the total number of velocity increments in the total velocity record of the length $\lambda = Nl = Nnr$. The length λ is presumed to be sufficiently large. For even orders, as follows from (7), there is no difference between $D_{2Gp}(l, r)$, $D_{Gp}(r)$, and $D_{Kp}(r)$, i.e. $D_{2Gp}(l, r) \equiv D_{Gp}(r) \equiv D_{Kp}(r)$. However, for odd orders the situation is different. Kolmogorov's and generalized structure functions of odd orders follow from (7) as special (extreme) cases of $D_{2Gp}(l, r)$, i.e.

$$D_{2Gp}(l = r, r) \equiv D_{Gp}(r) \qquad \text{when} \quad l = r \tag{8}$$

and

$$D_{2Gp}(l = \lambda, r) \equiv |D_{Kp}(r)| \qquad \text{when} \quad l = \lambda. \tag{9}$$

From (7) it also immediately follows that $D_{Gp}(r) > |D_{Kp}(r)|$ for the odd p, or, in general, $D_{Gp}(r) \geqslant |D_{Kp}(r)|$ for any p.

The behaviour of $D_{2Gp}(l, r)$ in the range $r < l < \lambda$ is not clear but may potentially provide important information on turbulence structure, additional to that given by $D_{Gp}(r)$ and $D_{Kp}(r)$. Here we consider this problem at the heuristic level only, i.e. using some reasonable assumptions and scaling considerations. For a fixed r from the inertial subrange (i.e. $r \ll L_0$) one should expect that at small l the function $D_{2Gp}(l, r)$ will decrease with increase in l. This is because of the cancellation effect of negative and positive contributions to the internal sums in (7). Also, for such l a jth internal sum $\{\sum_i^{n(l)} \Delta u(r)_i^p\}_j$ can be either positive or negative. Further, at sufficiently large l the cancellation effect should saturate and all the internal sums in (7) will have the same sign. This leads to the following relationship:

$$D_{2Gp}(l, r) = \frac{1}{Nn} \sum_j^{N(l)} \left| \sum_i^{n(l)} \Delta u(r)_i^p \right|_j = \frac{1}{Nn} \left| \sum_j^{N(l)} \left\{ \sum_i^{n(l)} \Delta u(r)_i^p \right\}_j \right| = |D_{Kp}(r)|. \tag{10}$$

Two regimes of $D_{2Gp}(l, r)$ can be distinguished from the above considerations: (i) a small-scale range of l where $D_{2Gp}(l, r)$ decreases with increase in l; and (ii) a large-scale range where $D_{2Gp}(l, r)$ does not depend on l and $D_{2Gp}(l, r) \equiv |D_{Kp}(r)|$. There should be a characteristic scale L that separates these two regimes. The scale L may depend on p and is clearly different from the integral turbulence scale L_0 as it characterizes long-term correlations between velocity increments. Thus, our first hypothesis assumes the existence of regimes (i) and (ii), and the scale L. Our second assumption is about regime (i). We postulate that in the range $r \leqslant l \leqslant L$ the function $D_{2Gp}(l, r)$ demonstrates scaling behaviour, i.e.

$$D_{2Gp}(l, r) \propto l^{-k(p)} \tag{11}$$

where the scaling exponents $k(p)$ may be different for different p. Combining (10) and (11) one can obtain:

$$\chi_p(l, r) = \frac{D_{2Gp}(l, r)}{D_{Gp}(r)} = \frac{\sum_j^{N(l)} |\sum_i^{n(l)} \Delta u(r)_i^p|_j}{|\sum_i^{N(r)} \Delta u(r)_i^p|} = \left(\frac{l}{L}\right)^{-k(p)} = \left(\frac{r}{L}\right)^{-k(p)} \left(\frac{l}{r}\right)^{-k(p)} \tag{12}$$

where $\chi_p(l, r)$ is the normalized two-scale structure function. For $l = r$ we get from (12):

$$\sum_i^{N(r)} |\Delta u(r)^p|_i = \sum_i^{N(r)} |\Delta u(r)|_i^p = \left(\frac{r}{L}\right)^{-k(p)} \left| \sum_i^{N(r)} \Delta u(r)_i^p \right|. \tag{13}$$

Bearing in mind that $D_{Gp}(r) = (1/N_r) \sum_i^{N(r)} |\Delta u(r)|_i^p$ and $|D_{Kp}(r)| = (1/N_r)| \sum_i^{N(r)} \Delta u(r)_i^p|$, we can replace (13) by a relationship:

$$D_{Gp}(r) = \left(\frac{r}{L}\right)^{-k(p)} |D_{Kp}(r)| \tag{14}$$

which connects Kolmogorov's and generalized structure functions. Relationship (14) requires L to be specified. In general, the scale L may depend on r and p. At this point we are interested in the r-dependence of L rather than in its p-dependence. To satisfy dimensions we have to consider two options for the r-dependence of the scale L: (1) $L(p) \propto r$, i.e., $(r/L)^{-k(p)} = A_p =$ const does not depend on r (though may depend on p) and, therefore, $\xi^*(p) = \xi^{**}(p) = \xi^{***}(p)$; and (2) $L =$ const (i.e., L does not depend on r) that gives from (5) and (6): $\xi^{***}(p) = \xi^*(p) - k(p) = [1 - k(3)]\xi^{**}(p)$. The first option is much more realistic for r from the inertial subrange because of the assumed self-similarity of the velocity increments. Thus, we have from (12) and (14):

$$\chi_p(l,r) = A_p(l/r)^{-k(p)} = \chi_p(l/r) \qquad \text{for} \quad l < L \tag{15}$$

and

$$D_{Gp}(r) = A_p|D_{Kp}(r)| \tag{16}$$

where $A_p = \gamma(p)^{k(p)}$, and $\gamma(p) = L(p)/r$ are some (possibly universal) constants.

Finally, the above considerations establish the following relationships between $D_{2Gp}(l,r)$, $D_{Kp}(r)$, and $D_{Gp}(r)$:

$$D_{2Gp}(l,r) = D_{Gp}(r) = \left(\frac{r}{L}\right)^{-k(p)} |D_{Kp}(r)| \qquad \text{when} \quad l = r \tag{17}$$

and

$$D_{2Gp}(l,r) = \left(\frac{r}{L}\right)^{k(p)} D_{Gp}(r) = |D_{Kp}(r)| \qquad \text{when} \quad l \geqslant L. \tag{18}$$

3. Application

To test relationships (15) and (16) we have used the most reliable data from our data set measured in the Balmoral Irrigation Canal (North Canterbury, New Zealand). The experimental section was chosen 500 m downstream of the intake and about 350 m above the sediment pond. The cross-sectional shape of the channel is close to trapezoidal with top width of 6.2–7.0 m and bottom width of 3.5–4.5 m. To minimize side-wall effects, all measurements were in the central part of the flow. The main hydraulic parameters for the experiments were: flow rate $Q = 5.14$ m^3 s^{-1}; cross-sectional mean velocity $U_a = 1.05$ m s^{-1}; cross-sectional mean depth $H_a = 0.78$ m; hydraulic radius $R = 0.70$ m; depth at the measuring vertical $H = 1.05$ m; global Reynolds number Re $= U_a R/v = 0.74 \times 10^6$; global Froude number $Fr = U_a/\sqrt{gR} = 0.40$; and friction velocity $u_* = 6.94$ cm s^{-1} (obtained from the Reynolds stress measurements). The measurements were conducted using 3D acoustic Doppler velocimeters (ADV) with the sampling volume 10 cm beneath the transducer (Kraus *et al* 1994). The duration of point measurements was 2–20 min with a sampling interval of 0.04 s. The experimental procedure and data analysis are described in detail in Nikora and Goring (1998a). In our following considerations we use Taylor's frozen turbulence hypothesis which has been shown to be valid at distances from the bed larger than 10 cm (Nikora and Goring 1998b). In this paper we report only data obtained from the longest, 20 min, measurements conducted at 48–50 cm from the bed and 50–52 cm from the water surface. In our analysis we considered the longitudinal structure functions for longitudinal (downstream) velocities.

Figure 1 shows typical examples of the third-order Kolmogorov's structure function normalized with the spatial lag r. For spatial lags less than $\approx$50 cm the function $|D_{K3}(r)|/r$ is approximately constant, which is indicative of the inertial subrange (as would be expected from (4), i.e. $|D_{K3}(r)|/r = (\frac{4}{5})\bar{\varepsilon}$, Frisch (1995)). Also, the upper limit of the inertial subrange is close to the distance from the bed, in agreement with Yaglom (1993). Subsequently, the normalized two-scale structure function $\chi_p(l/r)$, relationship (12), has been calculated for the spatial lags r less than 50 cm and for $p = 3, 5, 7,$ and 9. Figure 2 shows typical graphs of $\chi_p(l/r)$

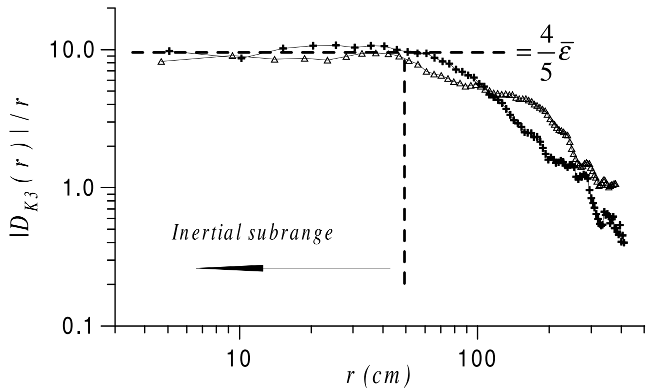

Figure 1. The normalized third-order Kolmogorov's structure function $|D_{K3}(r)|/r = f(r)$. The flat part of the graph at small r is indicative of the inertial subrange.

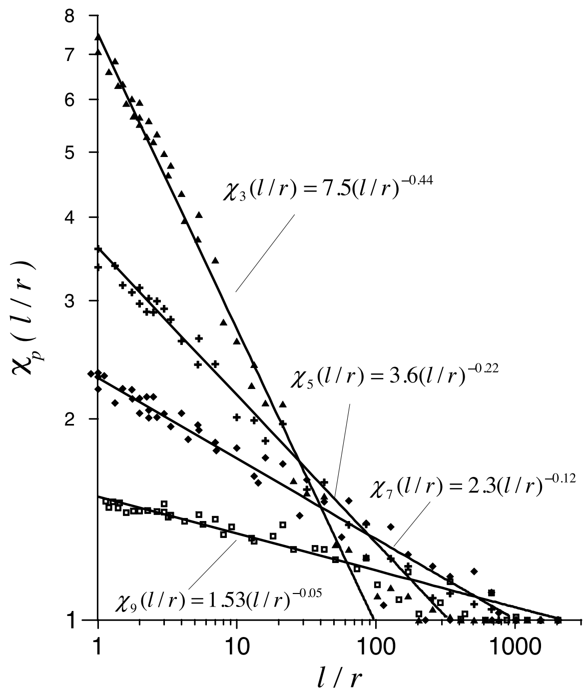

Figure 2. The function $\chi_p(l/r)$ for $p = 3, 5, 7$, and 9. Note how experimental points for different r from the inertial subrange collapse around single lines corresponding to different p. The slopes of the lines define the scaling exponents: $k(3) = 0.44$; $k(5) = 0.22$; $k(7) = 0.12$; and $k(9) = 0.05$.

which can be approximated as $\chi_p(l/r) = A_p(l/r)^{-k(p)}$ where $A_p = (r/L)^{-k(p)}$, in agreement with (15). In figure 3 we compare relationship (16) (using A_p from figure 2) with measured Kolmogorov's and generalized structure functions. As one can see, the agreement is quite satisfactory. Finally, figure 4 shows a relationship between $\gamma(p)$ and p, which confirms the assumed dependence of γ on p, and gives, approximately, $\gamma(p) = 3.8p^{2.88}$. This relationship as well as the reported scaling exponents $k(p)$ (figure 2) should be taken as a first approximation. The value of A_3 in our measurements (figure 3) appeared to be close to those which can be

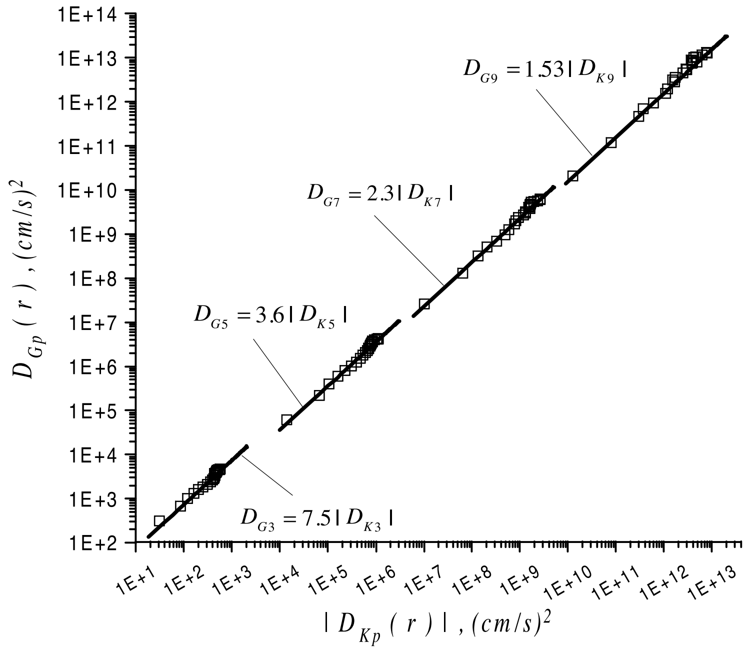

Figure 3. Comparison of measured Kolmogorov's and generalized structure functions for $p = 3$, 5, 7, and 9 with relationship (16) (A_p is taken from figure 2).

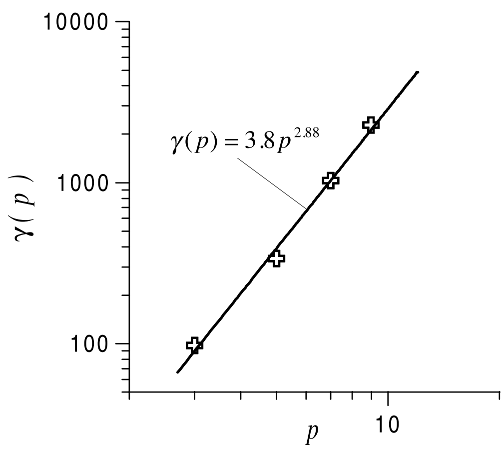

Figure 4. Relationship between γ and p.

extracted from data presented for the third-order structure functions in Benzi *et al* (1993b) and Herweijer (1995). This agreement suggests that the exponents $k(p)$ and $\gamma(p)$ may be universal. However, more tests are necessary to confirm this suggestion.

4. Conclusion

In this paper we have suggested and tested the relationship between Kolmogorov's and generalized structure functions. We argue that within the inertial subrange they are connected linearly, which implies identity between scaling exponents obtained from $D_{Kp}(r)$ and $D_{Gp}(r)$. Small differences between these exponents reported in Vainshtein and Sreenivasan (1994), Herweijer (1995), Arneodo *et al* (1996), and Sreenivasan and Dhurva (1998) are most probably due to statistical variability. We also suggest that more information about turbulence structure can be extracted using the two-scale structure function (7). The physical interpretation of the exponents $k(p)$ introduced in this paper still needs to be developed. We believe that our approach is of general character and may be useful in fields other than turbulence.

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